The Fueter Theorem and Dirac symmetries

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(partially joint work with V. Souček and P. Van Lancker)
Define differential operators on $\mathbb{R}^m$ (multivariate analysis)
Generate solutions of a special form (the Fueter theorem)
Proof by means of (Lie) symmetries of the operator
Explore the underlying algebraic framework
Find connections with special functions
Quaternionic analysis in $\mathbb{H}$: notations

- **Regularity** in quaternionic analysis $D_q f = 0$ with
  
  $$f : \mathbb{R}^4 \to \mathbb{H} : (t; x, y, z) \mapsto f(t; x, y, z),$$

  where $f = f_0 + i f_1 + j f_2 + k f_3$ (and $i^2 = j^2 = k^2 = -1$)

- **The Fueter-operator** on $\mathbb{H}$-valued functions:
  
  $$D_q := \partial_t + i \partial_x + j \partial_y + k \partial_z$$

- Quaternions can be written in ‘polar form’ as
  
  $$q := t + (ix + jy + kz) = t + \rho \omega \quad (\omega \in S^2 \subset \mathbb{R}^3)$$

  where $\rho^2 = x^2 + y^2 + z^2$ (Euclidean spatial squared norm)

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Recent interest in quaternionic analysis

- Applications in theoretical physics (Frenkel - Libine)
  → Schrödinger model for minimal representations of $O(3, 3)$
  → Quaternionic analysis, representation theory and physics

- Functional calculus and spectral analysis
  (Alpay - Colombo - Sabadini - Struppa - ...)

- Quaternionic Fourier transforms
  (Bujack - De Bie - Hitzer - Sangwine - Scheuermann)

- Relation with the conformal algebra: $\mathbb{R}_{1,3} \cong M^{2 \times 2}(\mathbb{H})$
Fueter’s original result (1935)

- Construction of regular functions from holomorphic ones:
  \[ \mathcal{F} : \ker \bar{\partial}_z \rightarrow \ker D_q \]

- **Input:** a holomorphic function on \( \Omega \subset \mathbb{C} \)
  \[ f(z) = u(x, y) + iv(x, y) \in \ker(\partial_x + i\partial_y) \]

- **Output:** a \( q \)-regular function (solution for \( D_q \))
  \[ F(t; x, y, z) = \mathcal{F}[f] := \Delta_4(u(t, \rho) + \omega v(t, \rho)) \]
  → the map \( \mathcal{F} \) combines **substitution + Laplace operator**

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Clifford analysis in a nutshell

- Inspired by the (massless) **Dirac equation** on $\mathbb{R}^{1,3}$:
  \[ i\hbar \gamma^\mu \partial_\mu \psi(t; x) = 0 \]
  $\rightarrow$ matrices $\gamma^\mu$ generate the Clifford algebra $\mathbb{R}_{1,3}$

- **Clifford algebras** exist in greater generality:
  \[(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{R}_{p,q} \rightarrow \mathbb{R}_{p,q} \otimes \mathbb{C} = \mathbb{C}_{p+q} := \mathbb{C}_m \]

- Natural question: study the corresponding **‘Dirac operator’**
  \[ \mathbb{R}_{p,q} = \text{Alg}(e_1, \ldots, e_m) \rightarrow \partial_x := \sum_{j=1}^m e_j \partial_{x_j} \]
Defining relations for $\mathbb{C}_m$ (similar for $\mathbb{R}_{p,q}$):

$$e_a e_b + e_b e_a = \{e_a, e_b\} = -2\delta_{ab}$$

→ $m$ anti-commuting complex (or hyperbolic) units

Clifford analysis thus refines harmonic analysis:

$$\partial_x^2 = (e_1 \partial_{x_1} + \ldots + e_m \partial_{x_m})^2 = -\Delta_m$$

Study of functions with values in (subset of) $\mathbb{C}_m$

→ typical example: a real subalgebra such as $\mathbb{R}_{0,m}$

→ ‘best choice’ (algebraically speaking): a spinor space
Clifford analysis generalises complex analysis
→ algebra isomorphism $\mathbb{R}_{0,1} \cong \mathbb{C}$
→ Dirac operator on $\mathbb{R}^2$ factorises $\Delta_2$
→ analogue of Cauchy formula available (for all $m$)

Clifford analysis generalises quaternionic analysis
→ algebra isomorphism $\mathbb{R}_{0,2} \cong \mathbb{H}$
→ Dirac operator on $\mathbb{R}^4$ factorises $\Delta_4$

Obvious question 1: how to generalise Fueter’s theorem? (i.e. special Dirac solutions from holomorphic functions)

Obvious question 2: do they have a special meaning?
Answers can be found in...

► Work done by Sommen-Tao-Kou

\[ \mathcal{F} : \ker \bar{\partial}_z \to \ker \partial_x \]

\[ \to \text{first in } \textit{odd} \text{ dimensions } m \ (\textit{obvious’ generalisation}) \]

\[ \to \text{later in } \textit{even} \text{ dimensions } m \ (\text{Fourier multipliers}) \]

\[ \to \text{further generalisations (‘shifted versions’)} \]

► Essentially also \textbf{substitution} + \textbf{Laplace operator}

\[ \mathcal{F}[f](x_1, \ldots, x_m) = \Delta_{m^2}^{m-1} [u(\rho, x_m) + \omega e_m v(\rho, x_m)] \]

with \[ \sum_j x_j e_j = \rho \omega + x_m e_m \in \mathbb{R}^{m-1} \oplus \mathbb{R} \ (\text{branched splitting}) \]
Plan for the lecture

- Give an **alternative proof** using symmetries for $\partial_x$
- We are interested in an algebraic approach:
  → easier to see the *link with special functions*
  → allows a generalisation to *other invariant operators*
  → possible connection with *other quadratic algebras*
  → possible connection with *slice regularity*
Symmetries for a differential operator

Consider a differential operator

\[ D : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{V}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{V}) \]

One says: \( \delta_1 \in \text{End} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{V}) \) is a generalised symmetry if there also exists an operator \( \delta_2 \in \text{End} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{V}) \) such that

\[ D\delta_1 = \delta_2 D \quad \Rightarrow \quad \delta_1 \in \text{End ker } D \]

Special case: \( \delta_1 = \delta_2 \Rightarrow \) a classical symmetry for \( D \)

First order (generalised) symmetries span a Lie algebra [Miller]
Generalised symmetries for the Dirac operator

- (obvious) translation symmetry operators: $[\partial_{x_j}, \partial_x] = 0$
- rotational symmetry operators: $[dL(e_{ij}), \partial_x] = 0$

$$dL(e_{ij})f(x) := \left( x_i \partial_{x_j} - x_j \partial_{x_i} - \frac{1}{2} e_{ij} \right) f(x),$$

where $e_{ij}$ generates a rotation (‘anti-symmetric matrix’)

- generalised Euler symmetry ($E_x = r \partial_r = \sum_j x_j \partial_{x_j}$):

$$\partial_x (2E_x + m - 1) = (2E_x + m + 1) \partial_x$$

So far: symmetries belonging to $(so(m) \oplus \mathbb{R}) \oplus \mathbb{R}^m$

(you may recognise a parabolic Lie algebra/Euclidean motions)

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The (monogenic) Klein inversion operator is defined by means of
\[
\mathcal{I} : \ker \partial_x \rightarrow \ker \partial_x : f(x) \mapsto \mathcal{I}[f](x) := \frac{x}{|x|^m} f \left( \frac{x}{|x|^2} \right)
\]

→ easily verified that the following operator equality holds:
\[
\mathcal{I} \partial_x \mathcal{I} = |x|^2 \partial_x \Rightarrow \partial_x \mathcal{I} = -\mathcal{I} |x|^2 \partial_x = (\mathcal{I} |x|^2 \mathcal{I}) \mathcal{I} \partial_x
\]

→ gives rise to another class of generalised symmetries:
\[
\partial_x (\mathcal{I} \partial_{x_j} \mathcal{I}) = \frac{1}{|x|^2} (\mathcal{I} \partial_{x_j} \mathcal{I}) |x|^2 \partial_x \quad (1 \leq j \leq m)
\]
The full conformal picture

- We have found the following Lie algebra of symmetries:

\[
\text{Alg}(\mathcal{I} \partial x_j \mathcal{I}) \oplus \left( \text{Alg}(dL(e_{ij})) \oplus \mathbb{R}(2E_x + m - 1) \right) \oplus \text{Alg}(\partial x_j)
\]

- Defines a realisation for the algebra \( so(1, m + 1) \)

  e.g.  \[ [\partial x_j, \mathcal{I} \partial x_k \mathcal{I}] = \delta_{jk}(2E_x + m - 1) - dL(e_{jk}) \]

  \( \rightarrow \) explains where the conformal weight \( m - 1 \) comes from

  \( \rightarrow \) in abstract form: \( \mathbb{R}^m \oplus (so(m) \oplus \mathbb{R}) \oplus \mathbb{R}^m \) (1-graded)
Intermezzo: the Lie algebra $\mathfrak{sl}(2)$

- Classical matrix definition:
  
  $$ \mathfrak{sl}(2) := \{ M \in \mathbb{C}^{2 \times 2} : \text{trace}(M) = 0 \} $$

- Well-known vector space basis \{X, Y, H\} given by
  
  \[
  \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}
  \]

- Defines a Lie algebra for $[A, B] := AB - BA$ with
  
  $[X, Y] = H$ \quad $[H, X] = +2X$ \quad $[H, Y] = -2Y$
Verma modules

- Take a highest weight vector $v_h$ in $\ker X$
- Repeated action of $Y \in \mathfrak{sl}(2)$ creates a vector space

\[
\mathbb{V}_\lambda^\infty := \bigoplus_{j=0}^{\infty} Y^j[v_h]
\]

(provided we have that $Y^j[v_h] \neq 0$ for all $j \in \mathbb{Z}^+$)
- Each of these vectors is an eigenvector for $H \in \mathfrak{sl}(2)$

\[
H[Y^j[v_h]] = (\lambda - 2j)Y^j[v_h]
\]

- **Result:** a Verma module $\mathbb{V}_\lambda^\infty$ with $\lambda \in \mathbb{C}$ (with restrictions)

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Specific subalgebra plays a crucial role

Lemma

The Lie algebra $\mathfrak{sl}(2)$ can be realised as

$$\mathfrak{sl}(2) = \text{Alg}(\partial_{x_j}, -I\partial_{x_j} I, 1 - m - 2E_x) = \text{Alg}(X, Y, H)$$

Arbitrary polynomial in $(x_1, \ldots, x_{m-1})$ generates a Verma module:

$$\mathcal{V}_{1-m-2k}^\infty := \bigoplus_{j=0}^{\infty} (-1)^j (I\partial_{x_m}I)^j P_k(x_1, \ldots, x_{m-1})$$

→ ‘weight spaces’ expressed in terms of Gegenbauer polynomials
→ these Verma modules will serve as images of the Fueter map

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The (poly-monogenic) inversion operator is defined by means of

\[ \mathcal{I}_\alpha : f(x) \mapsto \mathcal{I}_\alpha[f](x) := \frac{x}{|x|^{m+\alpha}} f \left( \frac{x}{|x|^2} \right) \quad (\alpha \in \mathbb{C}) \]

→ typical case of interest: \( \alpha = -2\ell \) with \( \ell \in \mathbb{Z}^+ \)
→ mapping properties differ from Klein inversion (\( \alpha = 0 \))

\[ \mathcal{I}_{-2\ell} : \ker \partial_x^{2\ell+1} \rightarrow \ker \partial_x^{2\ell+1} \]

indeed: \( f(x) \mapsto |x|^{2\ell} \mathcal{I}[f](x) \) (extra factor \( |x|^{2\ell} \) to be killed)

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Still a copy of $\mathfrak{sl}(2)$ available

**Lemma**

The Lie algebra $\mathfrak{sl}(2)$ can be realised as

$$\mathfrak{sl}(2) = \text{Alg}(\partial_{x_j}, -\mathcal{I}_\alpha \partial_{x_j} \mathcal{I}_\alpha, 1 - m - \alpha - 2E_x)$$

Explicit formula for the ‘creation operator’:

$$\mathcal{I}_\alpha \partial_{x_m} \mathcal{I}_\alpha = -|x|^2 \partial_{x_m} + x_m (2E_x + \alpha + m - 1) + x \wedge e_m$$

In order to get a Fueter theorem we need two ingredients:

→ a substitution map $z = x + iy \mapsto \ ?$

→ the action of a Laplace power

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The substitution map

Earlier results based on the following:

\[ u + iv \mapsto u + (\omega e_m)v = \text{scalar} + (\omega e_m)\text{scalar} \]

We know that in the Clifford algebra one has that

\[ (\text{vector}) \times (\text{vector}) = \text{scalar} + \text{bivector} \]

This suggests the following:

\[ z = x + iy \mapsto (\bar{e}_m x) = -e_m \sum_{j=1}^{m} e_j x_j = x_m + \sum_{j=1}^{m-1} e_j e_m x_j \]
Invoking the conformal symmetries

The following can now easily be proved:

\[ (- |x|^2 \partial_{x_m} + x_m (2 \bar{\mathcal{E}}_x + 1) + x \wedge e_m)(\bar{e}_m x)^k = (k + 1)(\bar{e}_m x)^{k+1} \]

This leads to (choose the appropriate \( \alpha \in \mathbb{C} \)):

\[ (\mathcal{I}_{2-m} \partial_{x_m} \mathcal{I}_{2-m})^k[1] = k!(\bar{e}_m x)^k \]

\( \rightarrow \) we have that \( \mathcal{I}_{2-m} = \mathcal{I}_{-2\ell} \iff \ell = \frac{m-2}{2} \) (for \( m \) even)

\( \rightarrow \) the operator \( \mathcal{I}_{3-m} \) then preserves solutions for \( \partial_x^{1+2\ell} = \partial_x^{m-1} \)

\( \rightarrow \) so does the creation operator \( \mathcal{I}_{2-m} \partial_{x_m} \mathcal{I}_{2-m} \)

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Fueter’s theorem

We have now shown (for odd dimensions $m$):

$$\partial_x^{m-1}(\overline{e}_m x)^k = (-1)^{\frac{m}{2}-1} \partial_x \left( \Delta_x^{\frac{m}{2}-1}(\overline{e}_m x)^k \right) = 0$$

As this holds for all $k$, the result for holomorphic $f(z)$ follows:

$$f(z) \in \ker \overline{\partial}_z \mapsto \Delta_m f(\overline{e}_m x) \in \ker \partial_x$$

Explicit expressions for Fueter images available:

$$\mathcal{F}[z^{k+m-1}] \sim (\mathcal{I}_0 \partial_{x_m} \mathcal{I}_0)^k[1] \sim \pi_m \left( |x|^k C_k^{\frac{m}{2}-1}(t) \right)$$

with $t = \cos \theta = \frac{x_m}{|x|} \in [-1, +1]$ fixed by the choice of $e_m$

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The monogenic projection operator $\pi_m$

- Complex case: $\Delta_2 = 4\partial_z\bar{\partial}_z$, so

$$\Delta_2 f(z) = 0 \implies f(z) = f_0(z) + \bar{z}f_1(z) \xrightarrow{\pi_2} f_0(z)$$

with $\bar{\partial}_z f_0(z) = \bar{\partial}_z f_1(z) = 0$

- We know that $\partial_x^2 = -\Delta_m$, so

$$\Delta_x f(x) = 0 \implies f(x) = f_0(x) + xf_1(x) \xrightarrow{\pi_m} f_0(x)$$

with $\partial_x f_0(x) = \partial_x f_1(x) = 0$
The role of the Gegenbauer polynomials

- In a sense, Fueter’s theorem does the following:
  \[ \sum_{k \in \mathbb{Z}} c_k z^k \mapsto \sum_{k=0}^{\infty} \pi_m \left( C_{k,m} |x|^k + \frac{D_{k,m}}{|x|^{k+m-2}} \right) C_k^{m-1}(t) \]

- Gegenbauers arise as a consequence of branching rules (both for harmonics and monogenics on \( \mathbb{R}^m \))
  \[ \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m) \cap \ker \partial_x \cong \bigoplus_{j=0}^{k} \mathcal{P}_j(\mathbb{R}^{m-1}, \mathbb{C}_m) \cap \ker \partial_x \]

- Special role for \( j = 0 \): these are the Gegegenbauers

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Different interpretation for Fueter’s theorem

- Complex powers $z^k$ are mapped to special components for the restriction of homogeneous solutions for $\partial_x$ on $\mathbb{R}^m$ to $\mathbb{R}^{m-1}$
- The result is a slice monogenic function (Colombo et al.)
- **Obvious question:** can one restrict from $\mathbb{R}^m$ to $\mathbb{R}^{m-p}$ ($p > 1$)
  → leads to bi-axial Fueter theorems (Jacobi polynomials)
  → may lead to refinements of the slice theorems
Thank you for your attention!