Overview of Quaternion and Clifford Fourier Transformations

Eckhard Hitzер

Department of Material Science
International Christian University, Tokyo, Japan
http://erkenntnis.icu.ac.jp/

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Quote and Acknowledgments

Jesus Christ on World Peace:
"Peace I leave with you; my peace I give you.
I do not give to you as the world gives.
Do not let your hearts be troubled and do not be afraid."

Bible: Gospel of John, chp. 14, verse 27.

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CONTENTS

1. Clifford geometric algebra and calculus
   - Clifford geometric algebras over vector spaces $\mathbb{R}^{p,q}$
   - Basic Clifford geometric algebra based calculus

2. Overview of Clifford Fourier Transforms (CFT)
   - General geometric (Clifford) FT
   - Sommen+Bülow $Cl(0,n)$ CFT

3. Two-sided CFTs
   - Quaternion Fourier transform (QFT)
   - Generalizations of the QFT

4. One-sided CFTs
   - Spinor and pseudoscalar Clifford FTs
   - Clifford Linear Canonical Transforms

5. Conclusion and references
   - Conclusion
   - References and further information
Motivation 1: Complex Number Invariants [H. Li, 2008]

- Two complex numbers
  \[ x = x_1 + ix_2, \quad y = y_1 + iy_2 \]  
  have the product
  \[ xy = x_1 y_1 - x_2 y_2 + i(x_1 y_2 + x_2 y_1). \]  
- Product \( xy \) is not invariant under rotation in complex plain!
  \[ x \rightarrow xe^{i\theta}, \quad y \rightarrow ye^{i\theta}, \quad xy \rightarrow xy e^{2i\theta}. \]
- But \( \bar{xy} \) is invariant:
  \[ \bar{xy} \rightarrow \bar{x}e^{-i\theta}e^{i\theta}y = \bar{xy}. \]
  \[ \bar{xy} = (x_1 - ix_2)(y + iy_2) = \underbrace{x_1 y_1 + x_2 y_2} + i \underbrace{(x_1 y_2 - x_2 y_1)}, \]  
  (symmetric) inner product of vectors \( \vec{x} \cdot \vec{y} \)  
  \( \vec{x}, \vec{y} \) parallelogram area (anti-symmetric)
Motivation 2.1: Hamilton’s Quaternions

\[ \mathbb{C}l(0, 2) \cong \mathbb{C}l^+(3, 0) \cong \mathbb{C}l^+(0, 3) \]

- Gauss, Rodrigues and Hamilton’s 4D quaternion algebra \( \mathbb{H} \) over \( \mathbb{R} \) with 3 imaginary units:
  \[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \]  

- Quaternion
  \[ q = qr + qi i + qj j + qk k \in \mathbb{H}, \quad qr, qi, qj, qk \in \mathbb{R} \]  

- has quaternion conjugate (principle reverse in \( Cl^+_{3,0}, Cl(0, 2) \))
  \[ \bar{q} = qr - qi i - qj j - qk k, \]  

- Leads to norm of \( q \in \mathbb{H} \)
  \[ |q| = \sqrt{q\bar{q}} = \sqrt{qr^2 + qi^2 + qj^2 + qk^2}, \quad |pq| = |p||q|. \]  

- Scalar part
  \[ Sc(q) = qr = \frac{1}{2} (q + \bar{q}). \]  

- Inner product (defines orthogonality !)
  \[ Sc(p\bar{q}) = prqr + piqi + pjqj + pkqk \in \mathbb{R}. \]
Motivation 2.2: Pure Quaternions

- Two pure quaternions

\[ x = x_1 i + x_2 j + x_3 k, \quad y = y_1 i + y_2 j + y_3 k, \]  

have \( xy \) product

\[
xy = x_1 y_1 + x_2 y_2 + x_3 y_3 + (x_3 y_2 - x_2 y_3)i + (x_1 y_3 - x_3 y_1)j + (x_2 y_1 - x_1 y_2)k.
\]

- The symmetric part is the inner product of 3D vectors \( \vec{x} \cdot \vec{y} \).
- The antisymmetric part is the outer product of 3D vectors \( \vec{x}, \vec{y} \). Magnitude equals area of parallelogram spanned by \( \vec{x}, \vec{y} \).
- General outer product subject of Grassmann’s (1844) calculus of extension.
- Inner and outer product in \( nD \) setting were unified by Clifford (1878) in his associative geometric algebras \( Cl(p,q) = Cl_{p,q} \):

\[
ab = a \cdot b + a \wedge b, \quad \forall a, b \in \mathbb{R}^{p,q}.
\]
Example: $Cl_3$ Clifford geometric algebra (GA) of Euclidean space $\mathbb{R}^3$

- Orthonormal basis $\{e_1, e_2, e_3\}$ of 3-dim. Euclidean space $\mathbb{R}^3 = \mathbb{R}^{3,0}$ gives 8-dim. basis of $Cl_3 = Cl_{3,0}$

$$\{1, e_1, e_2, e_3, e_2e_3, e_3e_1, e_1e_2, i = e_1e_2e_3\}. \quad (13)$$

- $i$ ... unit trivector, i.e. oriented volume of unit cube, $i^2 = -1$.
- Even grade subalgebra $Cl_3^+$ isomorphic to quaternions $\mathbb{H}$ (Hamilton), prominent in virtual reality, aerospace applications, crystal texture (orientation) analysis, etc.

$$\{1, i, j, k\}. \quad (14)$$

- Therefore elements of $Cl_3^+$ rotors (rotation operators), rotating all vectors and multivectors of $Cl_3$

$$x \in Cl_3 : \quad x \rightarrow R^{-1}xR, \quad R \in Cl_3^+. \quad (15)$$
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Clifford Algebra: Multivectors, blades, reverse, scalar product

- A general element (multivector) $M \in Cl_{p,q}$ represents a collection of subspaces of different dimensions (grades) $k: 0 \leq k \leq n$: scalars, vectors, bi-vectors, . . . , pseudoscalars (Name: $k$-vector parts)

$$M = \sum_A M_A e_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n,$$

blade index $A \in$ powerset $\mathcal{P}(1, 2, \ldots, n)$, $2^n$ coefficients $M_A \in \mathbb{R}$.

- **Principle reverse** of $M \in Cl_{p,q}$ replaces complex / quaternion conjugation.

- The **scalar product** of two multivectors $M, \tilde{N} \in Cl_{p,q}$ is defined as

$$M \ast \tilde{N} = \langle M \tilde{N} \rangle = \langle M \tilde{N} \rangle_0 = \sum_A M_A N_A.$$

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Overview of Quaternion and Clifford Fourier Transformations
Clifford geometric algebras over vector spaces $\mathbb{R}^{p,q}$

Clifford Algebra: Modulus, blade subspace, pseudoscalar

- The **modulus** $|M|$ of a multivector $M \in Cl_{p,q}$ is defined as

$$|M|^2 = M \star \widetilde{M} = \sum_A M_A^2.$$  \hspace{1cm} (18)

This includes: magnitude, length, area, volume, hypervolume.

- Important for applications: $n = 2$ or $3 \text{ (mod 4)}$ **pseudoscalar** $i_n = e_1 e_2 \ldots e_n$

$$i_n^2 = -1.$$  \hspace{1cm} (19)

- Geometry: A **blade** $B$ describes a vector subspace (example: $e_1, e_2$-plane)

$$V_B = \{ \mathbf{x} \in \mathbb{R}^{p,q} | \mathbf{x} \wedge B = 0 \}, \text{ e.g. } B = e_1 \wedge e_2 \iff V_B = \{ \mathbf{x} | \mathbf{x} = x_1 e_1 + x_2 e_2 \}.$$  

Its dual blade

$$B^* = B i_n^{-1}, \text{ e.g. } B^* = e_1 e_2 i_3^{-1} = e_3, \quad V_{e_3} \perp e_1 e_2,$$

describes the **complementary** vector subspace $V_B^\perp$.

- Pseudoscalar $i_n$ **central** for $n = p + q = \text{odd}$, [but not for $n = \text{even}$!]

$$i_n M = M i_n, \quad \forall M \in Cl_{p,q}, \quad n = \text{ odd.}$$
Multivector functions

- **Multivector valued function** $f : \mathbb{R}^{p,q} \rightarrow Cl_{p,q}$, $p + q = n$, has $2^n$ blade components ($f_A : \mathbb{R}^{p,q} \rightarrow \mathbb{R}$)

  \[
  f(x) = \sum_{A} f_A(x) e_A. \tag{20}
  \]

- We define the **inner product** of $\mathbb{R}^{p,q} \rightarrow Cl_{p,q}$ functions $f, g$ by

  \[
  (f, g) = \int_{\mathbb{R}^{p,q}} f(x) \overline{g(x)} \, d^n x, \tag{21}
  \]

  and the $L^2(\mathbb{R}^{p,q}; Cl_{p,q})$-norm

  \[
  \|f\|^2 = \langle (f, f) \rangle, \tag{22}
  \]

  \[
  L^2(\mathbb{R}^{p,q}; Cl_{p,q}) = \{ f : \mathbb{R}^{p,q} \rightarrow Cl_{p,q} | \|f\| < \infty \}. \tag{23}
  \]
Basic Clifford geometric algebra based calculus

**Square roots of \(-1\) in Clifford algebras [21]**

- \(i \in \mathbb{C}\) in the transformation kernel \(e^{i\phi} = \cos \phi + i \sin \phi\) is replaced by a \(\sqrt{-1}\) in \(\text{Cl}(p, q)\). For example pseudoscalars \(i \rightarrow i_n \in \text{Cl}(n, 0), n = 2, 3 (\text{mod} \ 4)\).
- What other \(\sqrt{-1}\) are there in \(\text{Cl}(p, q)\)? Answer: [21].
- Example: In \(\text{Cl}(p, q), p + q = 2\), the \(\sqrt{-1}\) are \(i = b_1 e_1 + b_2 e_2 + \beta e_{12}\), with \(b_1, b_2, \beta \in \mathbb{R}\), and \(\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2\).
- Manifolds of \(\sqrt{-1}\) in: \(\text{Cl}(2, 0)\) (left), \(\text{Cl}(1, 1)\) (center), and \(\text{Cl}(0, 2) \cong \mathbb{H}\) (right).
Extensive overview of Clifford Fourier Transforms (CFT)
Overview of Clifford Fourier Transforms (CFT)

- Clifford Algebra $\sqrt{-1}$
- 2-sided CFT
- 1-sided CFT
- Quaternion Fourier
- Cl(0,n) basis vector CFT
- Clifford Fourier Transformations (CFT)
- Operator exponential CFT
- Space-time FT
- 2-sided (1-sided) quaternion FT (QFT)
- Spacetime FT
- Multivector Wavepackets
- Volume-time FT
- Quaternion Fourier
- Stieltjes Transform
- Fract. QFT
- 3D CFT $i_3 = e_1 e_2 e_3$
- Plane CFT $i_2 = e_1 e_2$
General geometric (Clifford) Fourier transform [5]

- **Generalizations:** in $e^{-ix\omega}$: $i \in \mathbb{C} \rightarrow \sqrt{-1} \in Cl(p, q)$, $-ix\omega \rightarrow -s(x, \omega)$ with $s(x, \omega)^2 < 0$.

- **Incorporates** most of previously known CFTs with the help of very general sets of left and right kernel factor products

$$\mathcal{F}_{GFT}\{h\}(\omega) = \int_{\mathbb{R}^{p',q'}} L(x, \omega)h(x)R(x, \omega)d^{n'}x, \quad L(x, \omega) = \prod_{s \in F_L} e^{-s(x, \omega)},$$

with $p' + q' = n'$, $F_L = \{s_1(x, \omega), \ldots, s_L(x, \omega)\}$ a set of mappings $\mathbb{R}^{p',q'} \times \mathbb{R}^{p',q'} \rightarrow \mathbb{T}^{p,q}$ into the manifold of real multiples of $\sqrt{-1}$ in $Cl(p, q)$.

- $R(x, \omega)$ is defined similarly, and $h : \mathbb{R}^{p',q'} \rightarrow Cl(p, q)$ is the multivector signal function.


Overview of Clifford Fourier Transforms (CFT)

Clifford Algebra $\sqrt{-1}$

Clifford Wavelets

Clifford Fourier Transformations (CFT)

Operator exponential CFT

Commutative CFT

Color image CFT

2-sided CFT

Cl(0,n) basis vector CFT

Spacetime FT

Multivector Wavepackets

Volume-time FT

2-sided (1-sided) quaternion FT (QFT)

Quaternion Fourier Mellin Transform

Fract. QFT

Quaternion Fourier Stieltjes Transform

2-sided CFT

General geometric FT

1-sided CFT

Spinorial CFT

Pseudoscalar CFT $n=2,3(\text{mod } 4)$

Plane CFT $i_2 = e_1 e_2$

3D CFT $i_3 = e_1 e_2 e_3$
Sommen+Bülow: $\text{Cl}(0, n)$ basis vector Clifford FT [32, 4]

$$\mathcal{F}_{SB}\{h\}(\omega) = \int_{\mathbb{R}^n} h(x) \prod_{k=1}^{n} e^{-2\pi x_k \omega_k e_k} d^n x,$$

(25)

where $x, \omega \in \mathbb{R}^n$ with components $x_k, \omega_k$, and $\{e_1, \ldots, e_k\}$ is an orthonormal basis of $\mathbb{R}^{0,n}$, $e_1 = \ldots = e_k = -1$, $h : \mathbb{R}^n \rightarrow \text{Cl}(0, n)$.

For $n = 2$ and real signals $h : \mathbb{R}^2 \rightarrow \mathbb{R}$, this transform is identical to the quaternionic FT (see later).


Overview of Clifford Fourier Transforms (CFT)

Operator exponential CFT

Clifford Algebra $\sqrt{-1}$

Clifford Wavelets

Clifford Fourier Transformations (CFT)

Cl(0,n) basis vector CFT

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Plane CFT $i_2 = e_1 e_2$

3D CFT $i_3 = e_1 e_2 e_3$

Quaternion Fourier Mellin Transform

Quaternion Fourier Stieltjes Transform

Fract. QFT

Isomorphism
Application: For generalized color image Fourier descriptors.

Phase correlation of color images: analogous to shift $\hat{g}(\omega) = \hat{f}(\omega)e^{i\omega \cdot \Delta}$, get shift $\Delta$ from color image CFT by score aggregation.

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Quaternion Fourier Mellin Transform

Isomorphism

Fract. QFT

Quaternion Fourier Stieltjes Transform

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Overview of Quaternion and Clifford Fourier Transformations
Main type: general *two sided CFT* [19]

One kernel factor on each side

\[
\mathcal{F}^{f,g}\{h\}(\omega) = \int_{\mathbb{R}^{p'},q'} e^{-fu(x,\omega)} h(x) e^{-gv(x,\omega)} d^{n'} x, \tag{26}
\]

- \(f, g\) two constant \(\sqrt{-1}\) in \(Cl(p, q)\)
- phase functions \(u, v: \mathbb{R}^{p'},q' \times \mathbb{R}^{p'},q' \to \mathbb{R}\)
- signal function \(h: \mathbb{R}^{p'},q' \to Cl(p, q)\)
- often \(\mathbb{R}^{p'},q' = \mathbb{R}^{p,q}\).

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Spinorial CFT

pseudoscalar CFT

$\frac{n=2,3}{n=2,3} \text{mod 4}$

Plane CFT

$\mathbf{i}_2 = \mathbf{e}_1 \mathbf{e}_2$

3D CFT

$\mathbf{i}_3 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$
Quaternion Fourier transform (QFT) [14, 20]

\[
\mathcal{F}^{f,g}\{h\}(\omega) = \int_{\mathbb{R}^2} e^{-f x_1 \omega_1} h(x) e^{-g x_2 \omega_2} d^2x, \quad h : \mathbb{R}^2 \rightarrow \mathbb{H}, \quad f, g \in \mathbb{H} : f^2 = g^2 = -1.
\]

- **Variants:** left or right kernel factors is dropped, or both are placed together at the right or left side, or rotor forms \((L = \tilde{R})\).
- It was first described by Ernst, et al, [10, pp. 307-308] (with \(f = i, g = j\)) for spectral analysis in two-dimensional nuclear magnetic resonance, suggesting to use the QFT as a method to independently adjust phase angles with respect to two frequency variables in two-dimensional spectroscopy.
- Later Ell [8] independently formulated and explored the QFT for the analysis of linear time-invariant systems of PDEs (next).
- The QFT was further applied by Bülow, et al [3] for image, video and texture analysis (see later).
- Sangwine et al [31, 2]: color image analysis and analysis of non-stationary improper complex signals, vector image processing, and quaternion polar signal representations (see later).
2D complex FT and QFT (Images: T. Bülow, thesis [3])

T. Bülow: Applications to 2D gray scale images. (Color images: Ell & Sangwine [9].) One component of each transform kernel for different frequency values $u, v$.

complex FT intrinsically 1D

QFT intrinsically 2D

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Overview of Quaternion and Clifford Fourier Transformations
Convenient ± split of quaternions [E.H.: AACA 2007, ICCA9]

\[ q = q_+ + q_- \]
\[ q_\pm = \frac{1}{2} (q \pm iqj) \]

(27)

\[ \Rightarrow \text{2 orthogonal 2D planes in } \mathbb{R}^4. \text{ For explicit forms of } q_\pm, \text{ see references (below).} \]

- **Consequence**: modulus identity \(|q|^2 = |q_-|^2 + |q_+|^2\).
- **Generalization**: \(i, j \rightarrow \text{any 2 pure unit quaternions } f, g: f^2 = g^2 = -1\).
- **NB**: Even \(f = g\) makes perfect sense. Identical to simplex/perplex split of Ell & Sangwine.


Quaternion Fourier transform (QFT)

Split of *two-sided* quaternion FT (QFT)

\[
\mathcal{F}_q \{ f \}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^2} e^{-ix_1 \omega_1} f(x) e^{-jx_2 \omega_2} d^2 x.
\] (28)

Linearity leads to

\[
\mathcal{F}_q \{ f \}(\omega) = \mathcal{F}_q \{ f_- + f_+ \}(\omega) = \mathcal{F}_q \{ f_- \}(\omega) + \mathcal{F}_q \{ f_+ \}(\omega).
\] (29)

Simple quasi-complex forms for QFT of \( f_\pm \)

The QFT of \( f_\pm \) split parts of quaternion function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) have simple quasi-complex forms

\[
\mathcal{F}_q \{ f_\pm \} = \int_{\mathbb{R}^2} f_\pm e^{-j(x_2 \omega_2 \mp x_1 \omega_1)} d^2 x = \int_{\mathbb{R}^2} e^{-i(x_1 \omega_1 \mp x_2 \omega_2)} f_\pm d^2 x.
\] (30)

Discrete and fast QFT: Pei, Ding, Chang (2001) [29]

Discrete QFT

\[
\mathcal{F}_{DQFT}\{f\}(\omega) = \sum_{x_1=0}^{M-1} \sum_{x_2=0}^{N-1} e^{-i x_1 \omega_1 / M} f(x) e^{-j x_2 \omega_2 / N}, \quad x_1, x_2 \in \mathbb{N}. \tag{31}
\]

Fast QFT computation OK.

Application to image processing [3, 14]

### Application: Image transformations

Computation of image stretches, reflections and rotations in QFT spectrum of image.

### Quaternionic Gaussian filter (GF)

- Minimal uncertainty
- Texture segmentation


Geometric interpretation [20] of QFT integrand $e^{-i x_1 \omega_1} h(x) e^{-j x_2 \omega_2}$

- **Local rotation** by phase angle $-(x_1 \omega_1 + x_2 \omega_2)$ of $h_-(x)$ in the two-dimensional $q_-$ plane, spanned by $\{i + j, 1 - ij\}$.

- **Local rotation** by phase angle $-(x_1 \omega_1 - x_2 \omega_2)$ of $h_+(x)$ in the two-dimensional $q_+$ plane, spanned by $\{i - j, 1 + ij\}$.

Geometry of $\pm$-split of $\mathbb{H}$ in $\mathbb{R}^4$ picture.

Example [9]: Assigning colors RGB to $q_i, q_j, q_k, (q_r = 0)$, simplex/perplex split

$$(f = g = (i + j + k)/\sqrt{3} \text{ gray line}).$$

Top left: **Original.** Top right: $q_-$-part (luminance).
Bottom left: **Sum.** Bottom right: $q_+$-part ($q_i \leftrightarrow q_j$) (chrominance).

QFT of color image [9], \( f = g = \) gray line.

This representation uses polar decomposition of the QFT:
\[
\mathcal{F} = \rho_{\mathcal{F}} \ e^{\theta_{\mathcal{F}}} \ n_{\mathcal{F}}.
\]

- Top left: Original.
- Top right: Modulus \( \rho_{\mathcal{F}} = |\mathcal{F}| \) in log scale.
- Bottom right: Phase angle \( \theta_{\mathcal{F}} \) (\( 0 = \) red, \( \pi/2 = \) green hue, \( \pi = \) cyan hue).
- Bottom left: Pure part (axis) of \( n_{\mathcal{F}} \) RGB coded.

Direct frequency domain filtering [9], $f = g = \text{gray line}$.

From left to right (vertical pairs):
1) Original and modulus $\rho_{\mathcal{F}}$.
2) Low-pass filtered.
3) Band-pass filtered.
4) High-pass filtered.

Applications of QFT (T. Bülow, thesis [3])

- The QFT allows superior two-dimensional texture segmentation.
- The QFT leads to omni-directional disparity estimation (e.g. for video frames).
- NB: In both cases intrinsic limitations of corresponding complex methods are overcome.
- Colour-Sensitive Edge Detection using Hypercomplex (QFT) Filters.
- Quaternion Wiener Deconvolution for Noise Robust Color Image Registration.


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Clifford Fourier Transformations (CFT)

Cl(0,n) basis vector CFT
Commutative CFT
Color image CFT
2-sided CFT

General geometric FT

Spacetime FT
Multivector Wavepackets
Volume-time FT

2-sided (1-sided) quaternion FT (QFT)

Quaternion Fourier Mellin Transform

Fract. QFT
Quaternion Fourier Stieltjes Transform

1-sided CFT
Spinorial CFT

Pseudoscalar CFT $n=2,3 \,(mod\,4)$

Plane CFT $i_2 = e_1 e_2$
3D CFT $i_3 = e_1 e_2 e_3$

Eckhard Hitzer
International Christian University

Overview of Quaternion and Clifford Fourier Transformations
Quaternion Fourier Mellin transform (QFMT) [22]

- Polar coordinates in $\mathbb{R}^2$ lead to
  \[
  \mathcal{F}_{QM}\{h\}(\nu, k) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} r^{-f\nu} h(r, \theta) e^{-gk\theta} d\theta dr / r, \quad \forall (\nu, k) \in \mathbb{R} \times \mathbb{Z},
  \]
  with quaternion valued signal $h : \mathbb{R}^2 \to \mathbb{H}$ such that $|h|$ is summable over $\mathbb{R}^*_+ \times S^1$ under the measure $d\theta dr / r$, $\mathbb{R}^*$ the multiplicative group of positive non-zero numbers, and $f, g \in \mathbb{H}$ two $\sqrt{-1}$.

- $\mathcal{F}_{QM}$ can characterize 2D shapes rotation, translation and scale invariant, possibly including object color and vector patterns encoded in the quaternionic components of $h$.

- The QFMT can be generalized straightforward to Clifford FMT applied to signals $h : \mathbb{R}^2 \to Cl(p, q), p + q = 2$, with two $\sqrt{-1}$: $f, g \in Cl(p, q), p + q = 2$.

- Case $Cl(1, 1)$ important for hyperbolic geometry and two-dimensional special relativity.


Comparison of complex FMT, Quat. FT and QFMT [22]

Figure 5. Left: Kernel of FMT. Right: Kernel of QFMT. $k, v \in \{0, 1, 2, 3, 4\}$.

Figure 6. Left: QFT kernel, right: QFMT kernel. Top row: $q_+$ parts: $1 + fg$ and $f - g$ components. Bottom row: $q_-$ parts: $1 - fg$ and $f + g$ components.
The QFMT splits 2D image in parts with radial/angular symmetries [22]

**Figure 7.** *Left:* Four components of the QFMT kernel \((v = 2, k = 3)\). *Right:* Symmetries of four components of the QFMT kernel \((v = 2, k = 3)\).
QFMT kernel: Radial/angular resolution tuning, scale invariance [22]

**Figure 8.** Left: High angular resolution. Center: High radial resolution. Right: High radial and angular resolution.

**Figure 9.** Illustration of QFMT scaling.
Overview of Clifford Fourier Transforms (CFT)

- Clifford Algebra $\sqrt{-1}$
- Clifford Wavelets
- Operator exponential CFT
- Clifford Fourier Transformations (CFT)
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Volume-time Fourier transform \([14, 17]\)

- The \(Cl(3, 1)\) volume-time subalgebra with basis \(\{1, e_t, i_3, i_{st}\} \cong \mathbb{H}\) and allows to generalize the two-sided QFT to a volume-time Fourier transform

\[
\mathcal{F}_{VT}\{h\}(\omega) = \int_{\mathbb{R}^{3,1}} e^{-e_t t \omega} h(x) e^{-i_3 \vec{x} \cdot \vec{\omega}} d^4x,
\]  

(33)

with \(x = t e_t + \vec{x} \in \mathbb{R}^{3,1}, \vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3,\)

\(\omega = \omega_t e_t + \vec{\omega} \in \mathbb{R}^{3,1}, \vec{\omega} = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3.\)

- The split (52) with \(f = e_t, g = i_3 = e_t^*\) becomes the spacetime split of special relativity

\[
h_{\pm} = \frac{1}{2} (1 \pm e_t h e_t^*).\]

(34)


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isomorphism

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The volume-time Fourier transform can indeed be applied to multivector signal functions valued in the whole spacetime algebra $h : \mathbb{R}^{3,1} \to Cl(3,1)$ [14, 17]

$$F_{ST}\{h\}(\omega) = \int_{\mathbb{R}^{3,1}} e^{-e_t t \omega_t} h(x) e^{-i_3 \bar{x} \cdot \bar{\omega}} d^4x. \tag{35}$$

The volume-time FT includes the pseudoscalar spatial Clifford FT of $Cl(3,0)$:

$$F_{PS}\{h\}(\bar{\omega}) = \int_{\mathbb{R}^3} h(x) e^{-i_3 \bar{x} \cdot \bar{\omega}} d^3x.$$

The split (34) applied to spacetime Fourier transform (35) leads to a multivector wavepacket analysis

$$F_{ST}\{h\}(\omega) = \int_{\mathbb{R}^{3,1}} h_+(x) e^{-i_3 (\bar{x} \cdot \bar{\omega} - t \omega_t)} d^4x + \int_{\mathbb{R}^{3,1}} h_-(x) e^{-i_3 (\bar{x} \cdot \bar{\omega} + t \omega_t)} d^4x,$$

in terms of right and left propagating spacetime multivector wave packets.

Application: stretches, reflections, rotations, acceleration, boost of space-time signal in spectrum of SFT.

E. Hitzer, Quaternion Fourier Transform on Quaternion Fields and Generalizations, AACA, 17 (2007), 497–517.
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Clifford Wavelets

Clifford Fourier Transformations (CFT)

Operator exponential CFT

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Quaternion FOURIER Mellin Transform

Fract. QFT

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Overview of Quaternion and Clifford Fourier Transformations
One-sided Clifford FTs [23]

**Relationship**: One-sided CFTs are obtained by setting one of the phase functions $u$ or $v$ to zero in the two-sided CFT (26).

**Definition (CFT with respect to one square root of $-1$)**

Let $f \in Cl(p, q), f^2 = -1$, be any square root of $-1$. The general one-sided Clifford Fourier transform (CFT) of $h \in L^1(\mathbb{R}^{p,q}; Cl(p,q))$, with respect to $f$ is

$$\mathcal{F}_f \{h\}(\omega) = \int_{\mathbb{R}^{p,q}} h(x) e^{-f u(x,\omega)} d^n x,$$

where $d^n x = dx_1 \ldots dx_n$, $x, \omega \in \mathbb{R}^{p,q}$, and $u : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

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Eckhard Hitzler

Overview of Quaternion and Clifford Fourier Transformations
Spinor Clifford FT [1]

- A recent discrete spinor CFT is used for edge and texture detection, where the signal is represented as a spinor and the $\sqrt{-1}$ is a local tangent bivector $B \in Cl(3,0)$ to the image intensity surface ($e_3$ is the intensity axis).
- Can be applied to Gaussian filtering.

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Spinor and pseudoscalar Clifford FTs

One-sided CFTs which use a single pseudoscalar [15]

- Well studied and applied \((i_n \text{ pseudoscalar}, i_n^2 = -1)\)

\[
\mathcal{F}_{PS}\{h\}(\omega) = \int_{\mathbb{R}^n} h(x) e^{-i_n x \cdot \omega} d^n x, \quad i_n = e_1 e_2 \ldots e_n, \quad n = 2, 3 \text{(mod 4)}, \tag{38}
\]

where \(h : \mathbb{R}^n \to Cl(n, 0)\), and \(\{e_1, e_2, \ldots, e_n\}\) is the orthonormal basis of \(\mathbb{R}^n\).

- Historically the special case of (38), \(n = 3\), was already introduced in 1990 [24] for the processing of electromagnetic fields.

- This same transform \((n = 3)\) was later applied [12] to two-dimensional images embedded in \(Cl(3, 0)\) to yield a two-dimensional analytic signal, and in image structure processing.

- Moreover, it \((n = 3)\) was successfully applied to three-dimensional vector field processing in [7, 6] with vector signal convolution based on Clifford’s full geometric product of vectors.

- For embedding one-dimensional signals in \(\mathbb{R}^2\), [12] considered in (38) the special case of \(n = 2\), and in [7, 6] this was also applied to the processing of two-dimensional vector fields.

- Recent applications of (38) with \(n = 2, 3\), to geographic information systems (GIS) and climate data can be found in [34, 33, 26].
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Plane CFT

$3D$ CFT $i_2 = e_1 e_2$

$3D$ CFT $i_3 = e_1 e_2 e_3$
Pseudoscalar CFT and complex FT, Discrete CFT, Fast CFT

- Case of pseudoscalar $Cl_3$-CFT.
- A multivector signal decomposition corresponds to 4 complex signals.
- Clifford FT can then be implemented by 4 complex FTs.
- This form of the pseudoscalar CFT leads to the discrete CFT using 4 discrete complex FTs.
- 4 fast FTs (FFT) can be used to implement a fast pseudoscalar CFT.
Spinor and pseudoscalar Clifford FTs

Application: Vector Pattern Matching (J. Ebling, G. Scheuermann [7, 6])

Fast $Cl(3,0)$ convolution in CFT Fourier domain of $3 \times 3 \times 3 = 3^3$ (red), $5^3$ (yellow), $8^3$ (green) rotation patterns with gas furnace chamber flow field.
Spinor and pseudoscalar Clifford FTs

Clifford FFT in Geographic Information Systems (GIS)

1. Get $f(x, y; t, w)$ and calculate $R$ by equation:
   $$f = R\overline{R}^{-1}$$

2. Change $R$ to exponent form by the equation:
   $$R = e^{J/2}$$
   and find out the angle $\phi$ of $\overline{R}$

3. Transform the template: $M' = RMR^{-1}$

4. Performing convolution operation:
   $$h(i, j) = \sum_{a=1}^{3} \sum_{b=1}^{3} g(a, b) f(i - a, j - b)$$

5. Use Clifford FFT to solved the problem speedily.
   Firstly, express field as follows:
   $$f = [a + i f_{13} a] + [f_{1} + f_{2} i f_{3} a] + [f_{2} + f_{3} i f_{1} a] + [f_{3} + f_{1} i f_{2} a]$$
   Then Clifford FFT of any field can be defined as:
   $$F_{G}(f) = F_{G}[a + i f_{13} a] + F_{G}[f_{1} + f_{2} i f_{3} a] + F_{G}[f_{2} + f_{3} i f_{1} a] + F_{G}[f_{3} + f_{1} i f_{2} a]$$

6. Output the correlation coefficient and classify field by the distribution of energy

Spatio-temporal raster and vector field data analysis.


Extensive overview: CFT + Clifford Linear Canonical Transforms (LCT) + QDFT
Generalizing to Clifford Linear Canonical Transforms (LCT)

- Real and complex linear canonical transforms parametrize a continuum of transforms, which include the Fourier, fractional Fourier, Laplace, fractional Laplace, Gauss-Weierstrass, Bargmann, Fresnel, and Lorentz transforms, as well as scaling operations.

- Generalization to (finite-extension, complex) Clifford LCTs from the $Cl(0, n)$ basis vector CFT of Buelow and Sommen.


- Hypercomplex LCT related to pseudoscalar CFT in $Cl(3, 0)$.


- Quaternionic LCT related to 2-sided QFT and fractional QFT.


- Further new constructions of Clifford LCTs in PMAMCM 2014 proceedings (Santorini/Greece, 2014).

- Further new constructions of Quaternion Domain FT (QDFT) in Group30 proceedings (Ghent/Belgium, 2014). Signal domain now $\mathbb{H}$, not only $\mathbb{R}^2$. 
Clifford Fourier transforms can apply the manifolds of $\sqrt{-1} \in Cl(p, q)$ to create a rich variety of new FTs.

History of just over 30 years. Major steps: $Cl(0, n)$ CFTs, then pseudoscalar CFTs, Quaternion FTs.

In the 90ies especially applications in electromagnetic fields/electronics and signal/image processing dominated.

This was followed by color image processing and most recently applications in Geographic Information Systems.

This presentation could only feature a part of the approaches in CFT research, and only a part of the applications. Omitted: operator exponential CFT approach [H. De Bie, et al], CFT for conformal geom. algebra. Regarding applications, e.g. CFT Fourier descriptor representations of shape [B. Rosenhahn, et al] was omitted.

Note that there are further types of Clifford algebra/analysis related integral transforms: Clifford wavelets, Clifford radon transforms, Clifford Hilbert transforms, ... which we did not discuss.

Generalization to Clifford Linear Canonical Transforms, and quaternion domain FT (QDFT).
References and further information


References and further information


[15] E. Hitzer, B. Mawardi, *Clifford Fourier Transf. on Multivector Fields and Unc. Princ. for Dim. $n = 2 \ (mod \ 4)$ and $n = 3 \ (mod \ 4)$*, P. Angles (ed.), AACA, 18(S3,4), (2008), 715–736.


References and further information


References and further information


## News about Clifford Geometric Algebra

**GA-Net**


**GA-Net Updates (blog)**

Immediate access to latest GA news: [http://gaupdate.wordpress.com/](http://gaupdate.wordpress.com/)

**AGACSE 2015, Barcelona/Spain, 27-31 Jul. 2015**


**Geometric_Algebra Group (Google)**

Forum for discussion of Geometric Algebra questions and concepts (P. Joot)

[http://groups.google.com/group/geometric_algebra](http://groups.google.com/group/geometric_algebra)

geometric_algebra@googlegroups.com

_Soli Deo Gloria_

Thank you for your attention! Questions?